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On the breakup of accelerating liquid drops

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An accelerating liquid drop, under the action of surface tension, is shown to be unstable to small disturbances above a first critical value of the Bond number. Both numerical and second-order asymptotic methods are employed in order to characterize the normal-mode response and the neutral-stable modes at larger values of the Bond number. The transient response of an initially spherical drop that is accelerated by the flow of an external gas is studied as an initial-value problem. A unified theory, that includes acceleration as well as aerodynamic effects, is presented in order to account for the complete dynamic range of Weber and Bond numbers. The results are compared with experimental observations that range from continuous vibration to irreversible aerodynamic distortion and unstable shattering.

1. Introduction

The purpose of this study is to provide a unified theory for the breakup of liquid drops that are accelerated by virtue of a surrounding gasdynamic flow field. The mechanism of droplet breakup is relevant to many physical processes. We cite as an example the erosion of supersonic aircraft surfaces by rain, where the size and shape of the drops at impact is determined by their transient response to the shock layer. Modern research into the mechanisms of droplet breakup has been in large part experimental, beginning perhaps with the work of Volynski (1948) and Hinze (1948), (1955). Their results, along with the more recent experimental results of Engle (1958), Hanson, Domich & Adams (1963), Anderson & Wolfe (1965), Reinecke & Waldman (1970) and Simpkins (1971), indicate heretofore unrelated regimes of dynamic response. This is because, even in the absence of viscous effects, the droplet is subject not only to deformation arising from a non-uniform surface pressure distribution, but to the possible simultaneous effect of instability due to acceleration. Nonetheless, a theory has been developed that serves to unify many of the experimental results made over the past several years and, in addition, reveals several features which have not been delineated previously. Hopefully, the latter results will provide a useful guide to more detailed experimentation.

Because of the rather complicated nature of this problem, the results will be presented in several self-contained parts. In the first part of our study, we report some basic results concerning the stability of an accelerating liquid sphere. These results, which are essential to a clear understanding of the remaining parts of this study, are developed from the relevant eigenvalue problem. In particular, the value of a lowest critical Bond number, below which the drop is stable to all small disturbances, is obtained by both numerical and higher order asymptotic methods. Further, the response to a initial disturbance is characterized in terms of the normal modes as being either stable, quasi-stable, or unstable.

Once the results of the eigenvalue problem have been established, we shall treat the initial-value problem with a view to describing the combined effects of aerodynamic as well as acceleration forces. We shall show that, below the lowest critical Bond number and above a critical value of the Weber number, the drop will break up owing to the effect of aerodynamic forces alone. Above the lowest critical Bond number, both aerodynamic and acceleration forces act to break up the drop, and a model will be presented to account for the combined effect of aerodynamic distortion and instability. We begin our study by considering the stability of an accelerating liquid drop.

2. The eigenvalue problem

The centre of mass of a liquid drop is assumed to move with constant acceleration $\mathbf{g} = \bar{g}\mathbf{k}$. (The use of bars is intended to denote dimensional quantities.) We consider a system of spherical co-ordinates $(\bar{r}, \theta, \phi)$, with associated unit vectors $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$, which has its origin fixed at the centre of mass and is oriented so that the unit vector, $\mathbf{k} = \mathbf{e}_r \cos\theta - \mathbf{e}_\theta \sin\theta$, associated with the polar axis is pointed in the direction of the acceleration. The fluid velocity and related droplet configuration are symmetric with respect to the latter axis, i.e. $\partial/\partial\phi = 0$. During the process of deformation, the liquid, which is assumed to be incompressible and invisid, undergoes potential motion in which the mass of the drop is preserved and the nonlinear terms in the momentum and kinematic equations are negligible. The droplet configuration is assumed to be, at all values of the time $\bar{t}$, slightly deformed from the unperturbed spherical shape, $\bar{r} = \bar{r}_0$. The formulation of the problem, as stated above, serves to define a surface pressure distribution on the exterior of the drop in terms of the internal hydrodynamic pressure and the surface tension stress (body forces arising from magnetic effects or gravity are not considered). Our aim is to establish certain results by studying the eigenvalue problem which derives from the above formulation. The connexion between these results, and the response of an initially spherical liquid drop to an external surface pressure distribution of arbitrary form, will be discussed subsequently. However, an understanding of the latter problem is greatly enhanced by first studying the eigenvalue problem. We therefore proceed to consider the stability of a nearly spherical liquid droplet undergoing constant acceleration.

The momentum equation

$$\bar{\rho}_l \frac{d\mathbf{u}}{d\bar{t}} = -\nabla\bar{\mathscr{P}}_l - \bar{\rho}_l \mathbf{g}, \qquad (1a)$$

where $\bar{\rho}_l$, $\bar{\mathscr{P}}_l$ and $\mathbf{u}$ are the liquid density, pressure and velocity, respectively, is integrated under the assumption that there exists a velocity potential $\bar{\phi}$ such that

$$\mathbf{u} = \nabla\bar{\phi}, \quad \nabla^2\bar{\phi} = 0. \tag{1b}$$

The surface of the drop, $\bar{r}_s(\theta, \bar{t})$, is described by the function $F(\bar{r}_s, \theta, \bar{t})$:

$$F = \bar{r}_s - [\bar{r}_0 + \bar{\eta}(\theta, \bar{t})] = 0, \tag{1c}$$

which is subject to the kinematic condition

$$\partial F/\partial \bar{t} + \mathbf{u} . \nabla F = 0. \tag{1d}$$

In addition, we require that the volume of the drop be conserved, and that the origin of the co-ordinate system remain fixed at the centre of mass. These constraints are expressed for small $\bar{\eta}/r_0$ by

$$\int \bar{\eta}\, d\bar{s} = 0, \tag{2a}$$

$$\int \bar{\eta} \cos\theta\, d\bar{s} = 0, \tag{2b}$$

where $d\bar{s}$ is the differential surface area.

At the surface of the drop, $\bar{r} = \bar{r}_s$, the liquid pressure $\bar{\mathscr{P}}_l$ is equivalent to the sum of an externally applied pressure $\bar{\mathscr{P}}_e$, and the interfacial pressure due to surface tension $\bar{\mathscr{P}}_T$,

$$\bar{\mathscr{P}}_l = \bar{\mathscr{P}}_e + \bar{\mathscr{P}}_T. \tag{2c}$$

In addition, we have by Newton's law

$$-\int \bar{\mathscr{P}}_e \mathbf{n}\, d\bar{s} = \mathbf{F} = M\mathbf{g}, \tag{2d}$$

where $\mathbf{n}$ is an outward unit vector normal to the surface, and M is the constant mass of the drop.

We proceed to define a perturbation parameter ϵ which corresponds to the amplitude of the surface disturbance, and in terms of which

$$\left.\begin{aligned} \bar{\eta} &= \epsilon\bar{\eta}^{(1)} + O(\epsilon^2), \\ \bar{\phi} &= \epsilon\bar{\phi}^{(1)} + O(\epsilon^2), \\ \bar{\mathscr{P}} &= \bar{\mathscr{P}}^{(0)} + \epsilon\bar{\mathscr{P}}^{(1)} + O(\epsilon^2), \\ \mathbf{F} &= \mathbf{F}^{(0)} + \epsilon\mathbf{F}^{(1)} + O(\epsilon^2). \end{aligned}\right\} \tag{3}$$

The appropriate linearized equations are derived in the appendix, equations (A 1)–(A 18). We introduce dimensionless quantities appropriate to the acceleration effect, namely,

$$\left.\begin{aligned} &r = \bar{r}/\bar{r}_0, \quad \eta = \bar{\eta}/\bar{r}_0, \quad \eta_s = \bar{r}_s/\bar{r}_0 = 1 + \eta, \\ &t = \bar{t}(\bar{g}/r_0)^{\frac{1}{2}}, \quad \phi = \bar{\phi}/(r_0^{\frac{3}{2}}\bar{g}^{\frac{1}{2}}), \quad Bo = \bar{g}\bar{\rho}_l r_0^2/\bar{\sigma}, \end{aligned}\right\} \tag{4}$$

where $\bar{\sigma}$ is the surface tension coefficient and Bo, the Bond number, is real and positive. The governing equations in dimensionless form are thus

$$\frac{\partial\phi^{(1)}}{\partial t} + \eta^{(1)}\cos\theta = \frac{1}{Bo}[2 + L]\eta^{(1)} - \mathscr{C}\cos\theta \tag{5a}$$

on $r = 1$, where

$$L = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}, \quad \mathscr{C} = -\int_0^\pi \eta^{(1)} P_2(\cos\theta)\sin\theta\, d\theta,$$

and

$$\frac{\partial\eta^{(1)}(\theta,t)}{\partial t} = \frac{\partial\phi^{(1)}(1,\theta,t)}{\partial r}, \tag{5b}$$

where

$$\nabla^2\phi^{(1)} = 0 \quad (r \leqslant 1), \tag{5c}$$

with the restriction

$$\int \eta^{(1)} ds = \int \eta^{(1)}\cos\theta\, ds = 0. \tag{5d}$$

A normal modes

We seek solutions to the above set of equations in the form

$$\eta^{(1)}(\theta,t) = \exp\{i\beta t\} F(\theta), \tag{6a}$$

$$\phi^{(1)}(r,\theta,t) = \exp\{i\beta t\}\,\Phi(r,\theta), \tag{6b}$$

where a normal mode is expressed by

$$F(\theta) = \sum_{n=2}^{\infty} A_n P_n(\cos\theta), \tag{7a}$$

$$\Phi(r,\theta) = i\beta \sum_{n=2}^{\infty} \frac{r^n}{n} A_n P_n(\cos\theta), \tag{7b}$$

$$A_0 = A_1 = 0. \tag{7c}$$

We note that (6) and (7) satisfy (5*b, c*) identically. Furthermore, (7*c*) satisfies the constraints expressed by (5*d*). Substitution of (6) and (7) into (5*a, b*) gives

$$-\beta^2 \sum_2^\infty \frac{A_n}{n} P_n + \sum_2^\infty A_n P_1 P_n = -\frac{1}{Bo}\Sigma(n-1)(n+2) A_n P_n - \mathscr{C}\exp\{-i\beta t\} P_1, \tag{8a}$$

$$\mathscr{C} = -\exp i\beta t \int_0^\pi P_2 \sin\theta \left(\sum_2^\infty A_n P_n\right) d\theta = -\tfrac{2}{5} A_2 \exp\{i\beta t\}. \tag{8b}$$

The second term of (8*a*) may be written as

$$\sum_2^\infty A_n P_1 P_n = \sum_2^\infty \frac{n+1}{2n+1} A_n P_{n+1} + \sum_2^\infty \frac{n}{2n+1} A_n P_{n-1}$$

$$= \sum_2^\infty \frac{n}{2n-1} A_{n-1} P_n + \sum_2^\infty \frac{n+1}{2n+3} A_{n+1} P_n + \tfrac{2}{5} A_2 P_1, \tag{9}$$

under the restriction $A_1 = 0$. We define the eigenvalue

$$\lambda = \beta^2, \tag{10a}$$

in terms of which (8*a*) becomes

$$\frac{n^2}{2n-1} A_{n-1} + \left[\frac{n(n-1)(n+2)}{Bo} - \lambda\right] A_n + \frac{n(n+1)}{2n+3} A_{n+1} = 0 \tag{10b}$$

for $n = 2, 3, 4, \ldots$, with $A_1 = 0$. It is important to note that the perturbed external surface pressure distribution $\mathscr{P}_e^{(1)}$, which gave rise to the term involving $A_2 P_1$ in (8), served to annihilate the only other term, (9), involving P_1. Had the pressure perturbation been set to zero, we would have obtained the result $A_2 = 0$, which, in combination with the requirement $A_1 = 0$ and the difference equation (10), would yield the self-consistent trivial solution

$$A_n = \eta^{(1)} = 0.$$

Further insight into the normal mode expansion may be gained by considering (A 15) in the limit $\bar{g} \rightarrow 0$. In this case, (5*a*) reduces to

$$\frac{\partial \phi^{(1)}}{\partial t} = \frac{1}{Bo}[2+L]\,\eta^{(1)}, \quad r = 1 \quad (Bo \ll 1). \tag{11a}$$

It may be verified by direct substitution into (11*a*) and (5*c*, *d*) that the normal modes for $Bo \ll 1$ are

$$\left.\begin{aligned} \eta^{(1)} &= A_n \exp\{i\beta_n t\}\, P_n(\cos\theta), \\ \phi^{(1)} &= i\beta_n A_n \frac{r^n}{n} \exp\{i\beta_n t\}\, P_n(\cos\theta), \end{aligned}\right\} \tag{11b}$$

where
$$\beta_n = \left[\frac{n(n-1)\,(n+2)}{Bo}\right]^{\frac{1}{2}}. \tag{11c}$$

Thus, in the case of free vibrations which are dominated by the effects of surface tension, the normal modes are expressed simply in terms of zonal harmonics. The characteristic frequency for such vibrations may also be recovered from (10*b*), i.e. $\lambda \rightarrow \beta_n^2$ as $Bo \rightarrow 0$. However, when surface tension and acceleration effects are of the same order, the normal modes are not the zonal harmonics and the formulation (6)–(10) is required. In the following sections we address ourselves to the solutions of (10).

3. The eigenvalues and neutral-stable modes

In general, we seek the eigenvalues λ of the difference equation (10) for positive real values of the Bond number *Bo*. Of particular interest is the infinite set of 'critical' Bond numbers $(Bo)_i$, each of which give rise to a zero eigenvalue $\lambda = 0$, referred to as a state of neutral stability. The associated eigenfunctions will be referred to as neutral-stable modes. The eigenvalue problem has been solved both by numerical and asymptotic methods. First, the eigenvalues of the difference equation (10) were found numerically, for given values of *Bo*, by truncating the coefficients above A_N for successively larger values of N, until the lower eigenvalues had converged. The mathematical justification for this procedure, as well as proofs that the eigenvalues λ and critical Bond numbers $(Bo)_i$ are real, may be found in Bell Laboratories Memorandum 71–1634–5. In order to check the numerical computation of the eigenvalues and to gain insight into the nature of the eigenfunctions, an asymptotic approximation to the critical Bond numbers and the corresponding neutral-stable modes has been obtained. The mathematical technique is detailed in Harper, Grube & Chang (1971); our aim here is to convey the physical meaning of the results.

For a given value of the Bond number, the corresponding infinite set of eigenvalues fall into one or more of the three distinguished states:

$$\lambda = \beta^2 > 0 \quad \text{(stable oscillation)},$$
$$\lambda = \beta^2 = 0 \quad \text{(neutral stability, critical Bond numbers)},$$
$$\lambda = \beta^2 < 0 \quad \text{(unstable growth)}.$$

The eigenvalues are bounded from below so that the number of negative λ, or unstable modes, is finite while the number of positive λ, or stable modes, is infinite. As the value of *Bo* increases, an increasing number of negative λ values occur. In particular, each time the increasing value of *Bo* exceeds a critical value $(Bo)_i$, another unstable mode appears. When the Bond number corresponds to a critical value $(Bo)_i$, there are $i-1$ unstable normal modes, one neutral-stable mode, the ith, and an infinite number of stable normal modes. Of particular interest is the value of the lowest critical Bond number

$$(Bo)_1 = 11{\cdot}22, \tag{12}$$

below which the drop is 'absolutely stable', i.e. stable to disturbances of arbitrary form. Such a circumstance obtains, as in a capillary tube, because geometric constraints limit the wavelength of the lowest disturbance mode.

The asymptotic approximation to the neutral-stable states was initiated by considering a second-order JWKB approximation to the steady-state version of (5) for large values of *Bo*. The latter approximation manifests a turning point at $\theta = \frac{1}{2}\pi$ and singular points at $\theta = 0, \pi$. The construction is completed by matching the JWKB approximation to boundary-layer approximations at the singular points, thereby determining the critical Bond numbers. The eigenfunctions, or neutral-stable modes consist of three-dimensional ring-like waves about the polar axis on the hemispherical surface, facing away from the direction of acceleration, and connected at the turning point to a smooth surface on the opposite hemispherical face. Near the polar axes, where the drop appears locally flat, the three-dimensional wave structure merges into a two-dimensional pattern with cylindrical symmetry. The equation for the critical Bond numbers is

$$(Bo)_i = \left[\frac{(i+\frac{1}{2})\pi + [(i+\frac{1}{2})^2\pi^2 - 8{\cdot}9]^{\frac{1}{2}}}{2{\cdot}4}\right]^2, \tag{13}$$

where the constant 8·9 represents a second-order correction. We note that the second-order correction, in conjunction with the definition of *Bo* as a real number, serves to initialize the index i at 1, thereby defining a lowest critical Bond number. Table 1 compares the first- and second-order approximations to the critical Bond numbers with the values obtained numerically. The second-order correction provides a significant adjustment at the lower modes and tends to confirm the numerical computations. Figure 1 displays the 1st, 5th and 9th neutral-stable modes obtained from the JWKB approximation.

For a given value of the Bond number, the eigenvalues and associated normal modes approach, for large values of λ, the form of (11). Table 2 compares the eigenvalues λ_n, computed numerically from (10), with values from (11*c*) at a Bond number $Bo = 20$. The plausibility of such behaviour rests with the fact that,

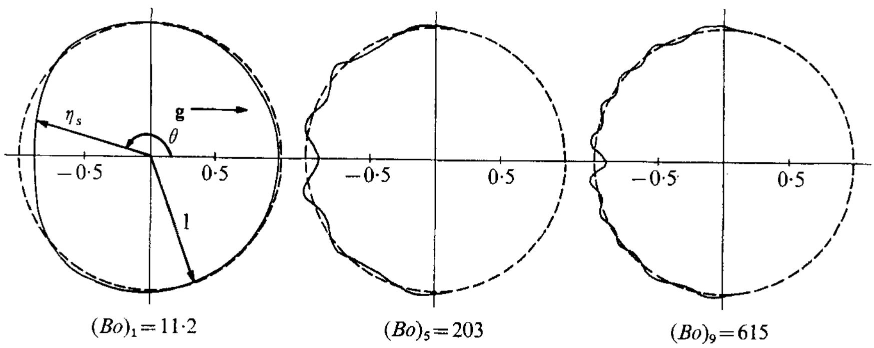

FIGURE 1. The neutral-stable modes of an accelerating liquid sphere.

Mode number i	First-order asymptotic approximation for large Bo	Second-order asymptotic approximation for large Bo	Numerical calculation
1	15·47	12·19	11·22
2	42·70	39·84	38·36
3	84·22	81·13	79·45
4	139·2	136·2	134·4
5	208·0	204·9	203·0
6	290·5	287·5	285·6
7	386·7	383·7	381·7
8	496·7	493·8	491·7
9	620·5	617·5	615·4
10	758·0	755·1	752·9

TABLE 1. The first ten critical Bond numbers $(Bo)_i$

n	Numerical computation, equation (10)	$\lambda_n = \beta_n^2 = n(n-1)(n+2)/Bo$
2	−0·72	0·40
3	1·21	1·50
4	3·54	3·60
5	6·97	7·00
6	11·98	12·00
7	18·89	18·90
8	27·99	28·00
9	39·59	39·60
10	53·99	54·00
11	71·49	71·50
12	92·39	92·40
13	116·99	117·00
14	145·60	145·60
15	178·50	178·50
16	216·00	216·00
17	258·40	258·40
18	306·00	306·00
19	359·10	359·10
20	418·00	418·00

TABLE 2. The first twenty eigenvalues λ_n for $Bo = 20$

for large wavenumbers n, the effects of surface tension are increased owing to the increased curvature, or, in terms of (5a),

$$\eta^{(1)} = O(1), \quad L\eta^{(1)} = O(n^2). \tag{14a}$$

Moreover, the associated high frequency of oscillation increases the inertia effect, or, in terms of (5a),

$$\phi^{(1)} = O(1) \quad \text{and} \quad \frac{\partial \phi^{(1)}}{\partial t} = O\left(\frac{\beta^2}{n}\right) = O(n^2). \tag{14b}$$

Thus, in the first approximation, the effects of acceleration are negligible and (5a) reduces to (11a).

When the Bond number exceeds the lowest critical value, the drop is unstable to an initial disturbance of arbitrary form. However, the drop is 'conditionally stable' to initial disturbances of a particular form. For example, if $Bo = 400$ (see table 1), the first seven normal modes are unstable. Under the condition that the initial disturbance was composed only of the higher, stable modes, the drop response would be stable. This characterization is important to the understanding of the initial-value problem, where the initial disturbance is most easily treated as an expansion in zonal harmonics. If the initial disturbance is a zonal harmonic of rather high order, then it is almost, but not exactly, a stable normal mode. However, such a disturbance will contain, as a component, an unstable normal mode of very small amplitude. In such a case we term the response 'quasi-stable' because only after a sufficiently long period of time will the instability be manifest.

The initial-value problem of determining the change in droplet shape with time is, in principle, solvable by means of a known set of eigenvalues and associated normal modes for each given value of the Bond number. However, such an approach would prove extremely difficult. Moreover, application of the results to the problem of an arbitrary external aerodynamic pressure distribution requires at least closed-form approximations, if not an exact solution. For these reasons the initial-value problem has been treated in terms of asymptotic expansions for small values of the time and for large wavenumber n. However, the results concerning the critical Bond number, absolute stability, and quasi-stable response, developed from the eigenvalue problem, will prove extremely important to an understanding of the initial-value problem.

4. The initial-value problem

In § 3 it was pointed out that the rate of unstable growth can be quite small if the Bond number is not sufficiently large, and such response was termed quasi-stable. The concept of the relative rate of unstable growth is important because, above a critical value of the Weber number, aerodynamic forces also act to deform the drop continuously and irreversibly. In order to make these concepts precise, we consider the transient response of an initially spherical drop that is suddenly accelerated by the flow of an external gas. This initial-value problem is appropriate, for example, to the interaction between a raindrop and the shock layer about a supersonic aircraft.

We begin by describing the results of an iterative procedure for calculating the response to an initial deformation in the form of a zonal harmonic. These results demonstrate the effect of Bond number and wavenumber on the unstable growth that is manifest on the windward face of the drop. We then obtain a closed-form asymptotic approximation, for large orders of the initial zonal harmonic, that is in excellent agreement with the results of iteration. The asymptotic solution provides for the expansion, in zonal harmonics, of an arbitrary initial distortion from the spherical shape. The initial distortion is then characterized in terms of the aerodynamic effect, which acts on a time scale which is of higher order than the time scale appropriate to the acceleration effect. The aerodynamic effect is to distort the drop algebraically in time, when the Weber number exceeds a critical value. An estimate of this critical value, which is based on a nonlinear analysis and which agrees favourably with experimental observations, reveals that all unstable (exponential) growth is accompanied by algebraic aerodynamic deformation. A composite expansion is presented which describes the selective unstable amplification, due to acceleration, of certain modes of the initial aerodynamic distortion. When the Bond number exceeds the so-called quasi-stable regime $Bo \approx 10^5$, this expansion displays an initial aerodynamic effect followed by an unstable shattering from the windward surface, which is in accord with the most recent observations of droplet breakup.

An iterative procedure

We construct a sequence of approximate solutions to (5*a–d*) by iteration, subject to the initial conditions

$$\left.\begin{aligned} \phi^{(1)}(r,\theta,0) &= \phi_0^{(1)} = 0, \\ \eta^{(1)}(\theta,0) &= \eta_0^{(1)} = A_N^0 P_N. \end{aligned}\right\} \tag{15}$$

The construction, which after *i*th iterations leads to the form

$$\eta_i^{(1)} = \sum_{N-i\geqslant 2}^{N+i} A_n^i(t)\, P_n(\cos\theta), \tag{16}$$

proceeds as follows.

From (5*a*) we obtain

$$\frac{\partial \phi_1^{(1)}(1,\theta,t)}{\partial t} = -A_N^0\left[\frac{N}{2N+1}P_{N-1}+\frac{N+1}{2N+1}P_{N+1}+\frac{(N-1)(N+2)}{Bo}P_N\right]-\mathscr{C}P_1, \tag{17a}$$

where

$$\mathscr{C}\left\{\begin{aligned} &= 0, \quad \text{if} \quad N > 2, \\ &= -\tfrac{2}{5}, \quad \text{if} \quad N = 2, \end{aligned}\right\} \tag{17b}$$

which may be integrated directly. The function that satisfies (5*c*) is thus

$$\begin{aligned} \phi_1^{(1)}(r,\theta,t) = -A_N^0 t\Big[&\frac{N}{2N+1}r^{N-1}P_{N-1}+\frac{N+1}{2N+1}r^{N+1}P_{N+1} \\ &+\frac{(N-1)(N+2)}{Bo}r^N P_N\Big] \quad (N>2), \end{aligned} \tag{18}$$

and application of (5*b*) yields the first iterate

$$\begin{aligned} \eta_1^{(1)} = A_N^0\Big\{P_N-\frac{t^2}{2}\Big[&\frac{N(N-1)}{2N+1}P_{N-1}+\frac{(N+1)^2}{2N+1}P_{N+1} \\ &+\frac{N(N-1)(N+2)}{Bo}P_N\Big]\Big\} \quad (N>2). \end{aligned} \tag{19}$$

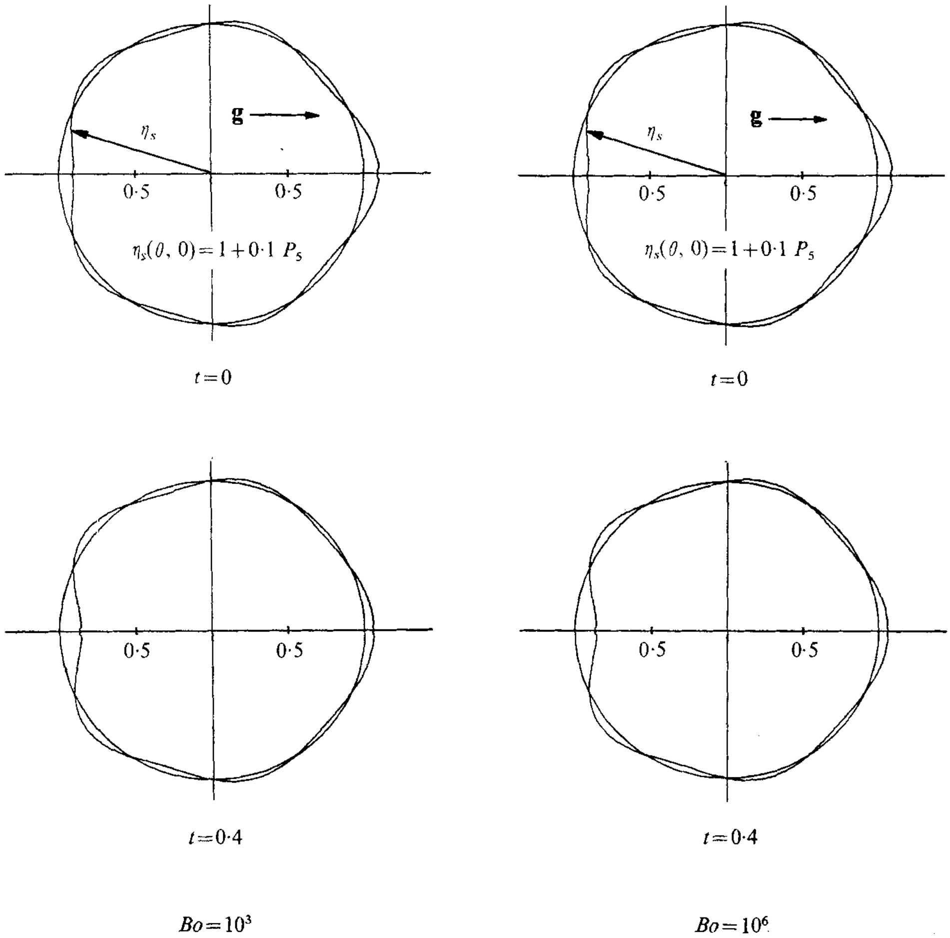

FIGURE 2. The effect of acceleration for an initial wavenumber of 5.

The second approximation is obtained by repeating the process. Furthermore, because ($5a$–d) are linear, each iteration reproduces its predecessor with a correction involving the next highest even power of t, and the last correction is obtained directly from the previous one by iteration. The result is an asymptotic expansion in even powers of the time. It is apparent that the initial state P_N produces a response involving all the zonal harmonics $P_n (n \geqslant 2)$; the two lowest harmonics P_0 and P_1 are not generated because of the presence of the factor $\mathscr{C}$ in ($17a$), and this fact serves to satisfy the constraint ($5d$).

The primary concern in this part of our study is to characterize the unstable response. From our study of the eigenvalue problem we concluded that, when the Bond number exceeds the lowest critical value 11·2, an initial disturbance in the form of a zonal harmonic P_n is unstable, because such a disturbance contains unstable normal modes as components. However, as was pointed out, for a given value of the Bond number zonal harmonics of sufficiently higher order are very nearly stable normal modes. In the case of such initial states, unstable behaviour is only manifest after a sufficiently long period of time, and we referred to the

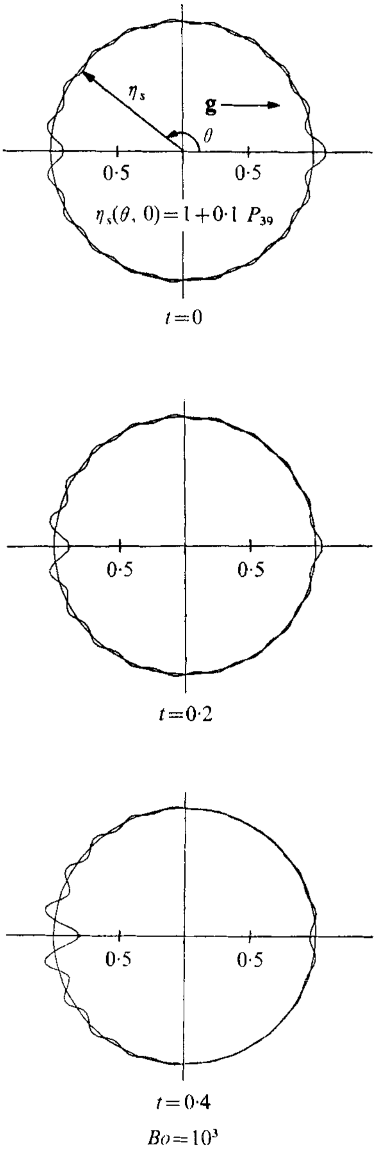

FIGURE 3. The effect of acceleration for an initial wavenumber of 19.

corresponding response as quasi-stable. It was also pointed out that, when the effect of acceleration is increased, a zonal harmonic of given order offers a poorer approximation to a stable normal mode and the response will be characterized by rapid exponential growth. These effects are illustrated in figures 2–4, which depict the free surface η_s calculated by the iterative procedure, to terms of $O(t^{20})$, for initial wavenumbers of 5, 19 and 39 at Bond numbers of 10^3 and 10^6. At an initial wavenumber of 5, the rate of unstable growth is very slow, while, at an initial wavenumber of 19, it is clearly manifest and predictably greater at the larger Bond number. At an initial wavenumber of 39 and a Bond number of 10^3, we

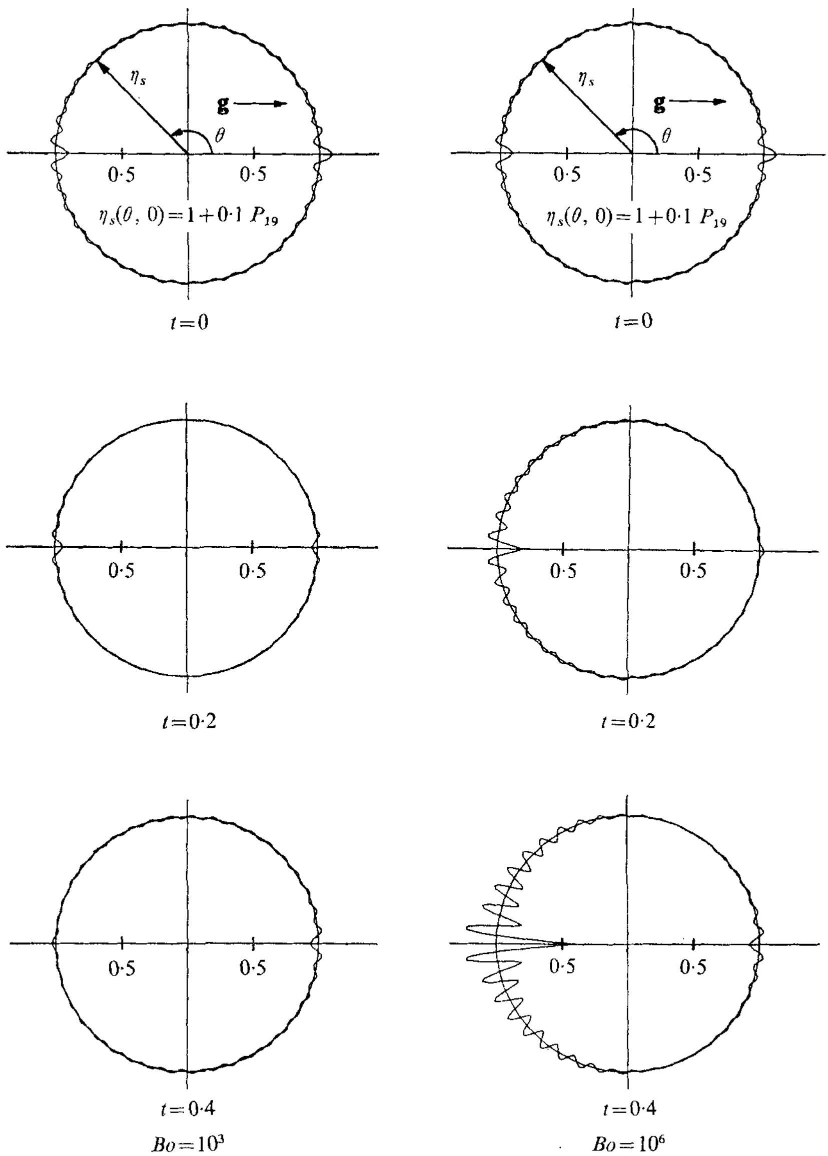

FIGURE 4. The effect of acceleration for an initial wavenumber of 39.

observe the quasi-stable response that occurs when the wavenumber is sufficiently large at a given value of the Bond number. The dependence on Bond number is evidenced by the fact that, when the Bond number is increased to 10^6 at the same initial wavenumber, the response is no longer quasi-stable. Further calculations indicate that quasi-stable response at a Bond number of 10^6 occurs when the initial wavenumber is about 10^3. One of the most notable aspects of the unstable response is that the exponential growth is manifest on the hemispherical surface facing away from the direction of acceleration.

We now address ourselves to the question of describing the response to an initial disturbance of arbitrary form. In view of the difficulties associated with the eigenfunction expansion, a normal mode analysis appears untenable. This would seem to suggest, as an alternative, a superposition of solutions obtained by applying the iteration procedure to each mode of the initial disturbance, represented as an expansion in zonal harmonics. However, an asymptotic approximation of closed form, which is in excellent agreement with the iteration procedure, has been found. The latter approximation, which is particularly well suited to describe the transient response of an initially spherical liquid drop to a suddenly applied external aerodynamic flow field, is detailed in the following sub-section.

An asymptotic approximation for large values of the wavenumber

When the initial surface disturbance P_N is of sufficiently high order, the variable coefficient appearing in (5*a*) plays an essentially different role in the operator L. When the wavenumber N is large, derivatives of $\eta^{(1)}$ and $\phi^{(1)}$ with respect to θ are large in magnitude, reflecting the effect of high curvature and particle velocity on surface tension and inertia forces. The variable coefficient, on the other hand, represents the acceleration effect that does not increase with wavenumber. We therefore follow Erdélyi (1968) and define the fast and slow variables δ and ζ, respectively. In terms of these new independent variables, we seek a solution of the form

$$\eta^{(1)} = C_1(\zeta, t) P_N(\delta), \tag{20a}$$

$$\phi^{(1)} = C_2(\zeta, t) r^N P_N(\delta), \tag{20b}$$

where the oscillations on the fast length scale δ are modulated on the slow length scale ζ. The differential equation (5*a*) is rewritten as

$$\frac{\partial \phi^{(1)}}{\partial t} + \eta^{(1)}\zeta = \frac{1}{Bo}[2+L]\,\eta^{(1)} - \mathscr{C}\zeta, \tag{21a}$$

where the operator appearing in (9*a*), as well as in the Laplacian (5*c*), denotes

$$L = \frac{\partial}{\partial \delta}(1-\delta^2)\frac{\partial}{\partial \delta}. \tag{21b}$$

The resulting solution is

$$C_1 = A_N \cos N^{\frac{1}{2}} \bar{\beta}_N t, \tag{22a}$$

$$C_2 = -A_N N^{-\frac{1}{2}} \bar{\beta}_N \sin N^{\frac{1}{2}} \bar{\beta}_N t, \tag{22b}$$

where

$$\bar{\beta}_N = \left[\zeta + \frac{(N-1)(N+2)}{Bo}\right]^{\frac{1}{2}} \tag{22c}$$

is $O(N)$ when $Bo = O(1)$ and is $O(1)$ when $Bo \geqslant O(N^2)$. In order to check the degree of approximation, we substitute the solution (22) into (5*a*–*d*). Clearly, (5*b*) is satisfied identically. When the leading terms of (5*a*, *c*) are scaled to be $O(1)$, it is found that (5*a*) is satisfied to $O(t^2 N/Bo\bar{\beta}_N^2)$, while (5*c*) is satisfied to $O(t^2/N\bar{\beta}_N^2)$. Another measure of the validity of the above approximation is to compare the expression for the free surface,

$$\eta_s = 1 + \epsilon A_N P_N(x) \cos\left\{N\left[x + \frac{(N-1)(N+2)}{Bo}\right]\right\}^{\frac{1}{2}} t, \tag{23}$$

with the result of iteration. It has been found that these approximations are in very close agreement over a wide range of wavenumbers, Bond numbers and values of the time. For example, all the results shown in figures 2–4 were reproduced by (23) to five significant figures at $t = 0{\cdot}2$, and to three significant figures at $t = 0{\cdot}4$.

The main feature displayed by (23) is that the drop is subject to unstable growth, on the hemispherical surface facing away from the direction of acceleration, for wavenumbers below a cut-off value N_C. For large values of the Bond number,

$$N_C \sim Bo + O(1). \tag{24}$$

The rate of exponential growth for the unstable modes is small near cut-off as well as for low wavenumbers. The maximum rate of growth occurs at a wavenumber $N_{\max}$,

$$N_{\max} \sim (Bo/3)^{\frac{1}{2}} + O(1), \tag{25}$$

or at approximately $0{\cdot}577\ N_C$, and is given by

$$\eta_{s\,\max} \sim 1 + \tfrac{1}{2}\epsilon P_{N_{\max}} \exp\{0{\cdot}62 Bo^{\frac{1}{4}} t\} \quad (x < 0). \tag{26}$$

The asymptotic approximation (23) provides a particularly useful means for computing the response to an initial disturbance of arbitrary form, as an expansion in zonal harmonics. Our ultimate goal is to describe the transient response of an initially spherical liquid drop to a suddenly applied external aerodynamic flow field. Such a circumstance is typified by the encounter between a raindrop and the shock layer about a supersonic aircraft. We must, therefore, address ourselves to a characterization of the initial excitation. As we shall see, the liquid droplet is subject to an initial distortion by virtue of the non-uniform distribution of surface pressure. Furthermore, the aerodynamic distortion occurs on a time scale of smaller order than the time scale appropriate to acceleration effects. This fact will ultimately allow for the construction of a composite approximation for the initial-value problem. In § 5 we detail the dynamic response of the liquid drop to external aerodynamic forces.

5. The aerodynamic effect

We begin by reformulating the equations of motion under the assumption that the drop is exposed to an external aerodynamic flow field of arbitrary form. It will be convenient to measure angular variations from the stagnation point on the windward surface by defining

$$\psi = \pi - \theta \quad \text{and} \quad \chi = \cos\psi. \tag{27a, b}$$

The asymptotic result (23) is simply modified by replacing the variable x with $-\chi$. We characterize the external gasdynamic pressure distribution $\bar{\mathscr{P}}_g$ in terms of the gas density $\bar{\rho}_g$ and the initial relative velocity U_∞:

$$\bar{\mathscr{P}}_g(\bar{r}, \psi, \bar{t}) = \tfrac{1}{2}\bar{\rho}_g U_\infty^2 \mathscr{P}_g(\bar{r}, \psi, \bar{t}), \tag{28a}$$

so that the external surface pressure $\bar{\mathscr{P}}_e$ is

$$\bar{\mathscr{P}}_e = \tfrac{1}{2}\bar{\rho}_g U_\infty^2 \mathscr{P}_g(\bar{r}_s, \psi, \bar{t}). \tag{28b}$$

The pressure distribution associated with the external gasdynamic flow field is assumed to be suddenly applied at the time $\bar{t} = 0$. The initial condition appropriate to a continuously applied pressure is

$$\bar{\phi}(\bar{r}, \theta, 0) = 0, \quad \bar{\eta}(\psi, 0) = r_0. \tag{29}$$

We now introduce dimensionless variables, based on a time scale t^*, appropriate to the aerodynamic effect:

$$\left.\begin{aligned} r &= \bar{r}/r_0, \quad \eta_1 = \bar{\eta}/r_0, \quad \eta_s = \bar{r}_s/r_0 = 1+\eta, \\ t^* &= \bar{t}U_\infty/r_0, \quad \phi^* = \bar{\phi}/(r_0 U_\infty). \end{aligned}\right\} \tag{30}$$

The parameters that appear in the differential equation are: the density ratio

$$\epsilon^* = \bar{\rho}_g/\bar{\rho}_l \ll 1, \tag{31a}$$

a dimensionless acceleration $\qquad g = \bar{g}r_0/U_\infty^2, \qquad (31b)$

a modified Weber number $\qquad We^* = \epsilon^{*-1}We, \qquad (31c)$

defined as the product of the inverse density ratio, and the Weber number

$$We = \bar{\rho}_g r_0 U_\infty^2/\bar{\sigma}. \tag{31d}$$

We seek a solution in the form of an expansion for small values of the density ratio ϵ^*,

$$\left.\begin{aligned} \eta &= \epsilon^*\eta^{(1)} + \epsilon^{*2}\eta^{(2)} + \ldots, \\ \phi^* &= \epsilon^*\phi^{(1)} + \epsilon^{*2}\phi^{(2)} + \ldots, \\ g &= \epsilon^* g^{(1)} + \epsilon^{*2} g^{(2)}(t^*)\ldots, \\ \mathscr{P}_g &= \mathscr{P}_g^{(0)} + \epsilon^*\mathscr{P}_g^{(1)}(t^*), \\ \mathscr{P}_e &= \mathscr{P}_e^{(0)} + \epsilon^*\mathscr{P}_e^{(1)}(t^*), \end{aligned}\right\} \tag{32}$$

and evaluate all functions at the surface of the drop in terms of a Taylor series about the original spherical surface. The perturbed gasdynamic pressure $\mathscr{P}_g^{(1)}(t^*)$ and surface pressure $\mathscr{P}_e^{(1)}(t^*)$ represent the effect of the deforming liquid surface, which alters the initial external flow field. Application of Newton's law (9*a*) yields the result

$$g^{(1)} = -\tfrac{3}{4}\int_0^\pi \mathscr{P}_e^{(0)} \sin\psi \cos\psi \, d\psi, \tag{33}$$

where $\qquad \mathscr{P}_e^{(0)} = \mathscr{P}_g^{(0)}(1, \psi, t^*). \qquad (34)$

The dimensionless acceleration is related to the conventional drag coefficient C_D by

$$g = \tfrac{3}{4}D/(\rho U_\infty^2 \pi r_0^2) = \epsilon^* \tfrac{3}{8} C_D, \tag{35a}$$

so that $\qquad g^{(1)} = O(1). \qquad (35b)$

The characteristic time for the aerodynamic effect,

$$T_{\text{aero}} = r_0/U_\infty, \tag{36a}$$

and the characteristic time for the acceleration effect,

$$T_{\text{accel}} = (r_0/\bar{g})^{\frac{1}{2}}, \tag{36b}$$

are not of the same order in the small parameter ϵ^*:

$$T_{\text{aero}}/T_{\text{accel}} = g^{\frac{1}{2}} \sim \epsilon^{*\frac{1}{2}} g^{(1)\frac{1}{2}}. \tag{36c}$$

Therefore, a time of $O(1)$ on the scale of aerodynamic deformation t^* represents a time of $O(\epsilon^{*\frac{1}{2}})$ on the scale t appropriate to instability:

$$t = g^{\frac{1}{2}} t^* \sim \epsilon^{*\frac{1}{2}} g^{(1)\frac{1}{2}} t^*. \tag{37}$$

The Weber number that appears as a parameter in the equations for the aerodynamic deformation is related to the Bond number

$$Bo = \bar{g}\bar{\rho}_l r_0^2/\bar{\sigma}, \tag{38}$$

which appears as a parameter in the equations appropriate to describe the instability, i.e. the effect of acceleration. In free flight, the relation between these parameters is

$$Bo = g We^* \sim \epsilon^* g^{(1)} We^* = g^{(1)} We. \tag{39}$$

We now examine the first-order approximation to the aerodynamic deformation in terms of the Weber number and drop the time scale t^*.

A linearized theory for the aerodynamic effect

In the first approximation, the problem becomes

$$\frac{\partial \phi^{(1)}}{\partial t^*} - \frac{1}{We^*}[2+L]\eta^{(1)} = -[g^{(1)}\chi + \tfrac{1}{2}\mathscr{P}_e^{(0)}] H(t^*) \tag{40a}$$

on $r = 1$, where We^* is assumed to be $O(1)$, and

$$L = \frac{\partial}{\partial \chi}(1-\chi^2)\frac{\partial}{\partial \chi}, \quad \chi = \cos\psi,$$

and $H(t^*)$ is the Heaviside step function.

$$\frac{\partial \eta^{(1)}}{\partial t^*} - \frac{\partial \phi^{(1)}}{\partial r} = 0 \quad (r = 1), \tag{40b}$$

where
$$\nabla^2 \phi^{(1)} = 0 \quad (r \leqslant 1), \tag{40c}$$

with the restriction
$$\int_{-1}^{1} \eta^{(1)} d\chi = \int_{-1}^{1} \eta^{(1)} \chi \, d\chi = 0, \tag{40d}$$

and the initial conditions

$$\phi^{(1)}(r, \psi, t^*) = \eta^{(1)}(\psi, t^*) = 0. \tag{40e}$$

We consider the first-order surface pressure to be a steady axisymmetric distribution of arbitrary form

$$\mathscr{P}_e^{(0)} = \sum_{1}^{\infty} \tfrac{1}{2}(2n+1)\, C_n P_n(\cos\psi), \tag{41a}$$

where $P_n(\chi)$ are the zonal harmonics, and where

$$C_n = \int_{-1}^{1} \mathscr{P}_e^{(0)} P_n \, d\chi. \tag{41b}$$

The first-order acceleration, which appears in (40*a*), is thus

$$g^{(1)} = -\tfrac{3}{4}C_1. \tag{42}$$

The assumption of a steady first-order surface pressure $\mathscr{P}_e^{(0)}$ is based on the estimate of Schlichting (1955) that boundary-layer separation on an impulsively started sphere is established in the characteristic time

$$t^* \approx 0{\cdot}39,$$

while the period of time over which the aerodynamic effect is relevant is

$$t^* = O(\epsilon^{*-\frac{1}{2}}).$$

The latter value is approximately thirty for an air-water drop interaction.

Comparison of (40*a*) and (1*a*) reveals that the dynamic response on the time scale t^* is quite different from that on the scale t. On the latter time scale, the effect of acceleration appears in the differential operator as a self-excitation $\eta^{(1)}x$ characteristic of Taylor instability. However, on the time scale t^*, the acceleration effect $g^{(1)}\chi$ appears as a forcing term that does not induce instability. Furthermore, to first order, the effect of acceleration is exactly balanced by the first mode of the aerodynamic surface pressure. That is, we may express the surface pressure distribution as

$$\mathscr{P}_e^{(0)} = -g^{(1)}\chi + \mathscr{P}_d^{(0)}, \tag{43a}$$

where $$\mathscr{P}_d^{(0)} = \sum_2^\infty \tfrac{1}{2}(2n+1)\,C_n P_n(\chi) = \sum_2^\infty K_n P_n. \tag{43b}$$

The first mode $-g^{(1)}\chi$ produces the rigid-body acceleration of the spherical drop which leads to the unstable amplification of small disturbances, on the time scale t, detailed in previous sections. The remaining pressure modes $\mathscr{P}_d^{(0)}$ serve to distort the drop on the time scale t^*, and in a co-ordinate system moving with the centre of mass. Equation (40*a*) may therefore be written

$$\frac{\partial\phi^{(1)}}{\partial t^*} - \frac{1}{We^*}[2+L]\,\eta^{(1)} = -\tfrac{1}{2}\mathscr{P}_d^{(0)}H(t^*). \tag{43c}$$

The solution to the above set of equations,

$$\phi^{(1)} = -\sum_2^\infty r^n \frac{(2n+1)C_n}{4\beta_n^*} P_n(\chi)\sin\beta_n^* t^*, \tag{44a}$$

$$\eta^{(1)} = \sum_2^\infty \frac{n(2n+1)C_n}{4\beta_n^{*2}} P_n(\chi)\,[\cos\beta_n^* t^* - 1], \tag{44b}$$

where $$\beta_n^* = [n(n-1)(n+2)]^{\frac{1}{2}}/We^*, \tag{44c}$$

indicates that, when $We^* = O(1)$ ($We = O(\epsilon^*)$), or less, the drop will vibrate in accord with the (dimensional) characteristic frequencies obtained from the normal mode expansion for $Bo \ll 1$, equation (11*c*). Our interest here, however, is to describe the response at moderate and large values of the Bond and Weber numbers. We note that, at a fixed value of the density ratio ϵ^*, the modal vibration amplitudes increase with increasing values of the Weber number. In the limit, as the modified Weber number approaches infinity, (44) reveals that the drop does not vibrate, but undergoes a continuous irreversible distortion that is algebraic in time:

$$\eta^{(1)} \sim \tfrac{1}{2}\epsilon^* t^{*2} \sum_{2}^{\infty} n(2n+1)\, C_n P_n(\chi) + O(\epsilon^{*2}) \tag{45}$$

as $W_e^* \to \infty$. If we consider the initial external gasdynamic field to be the symmetric unseparated flow about a sphere,

$$P_g^{(0)} = \left[\frac{1}{2r^2} + r\right] \cos\psi, \tag{46a}$$

(45) yields Burgers's (1958) estimate for the flattening of a raindrop shortly after collision with a shock wave

$$\eta_s = 1 - \tfrac{3}{4}\epsilon^* P_2(\chi) t^{*2} + O(\epsilon^{*2}). \tag{46b}$$

Equation (45) indicates that in the limit $W_e^* \to \infty$, i.e. as the ratio of surface tension to aerodynamic effects approaches zero, the drop will deform continuously and irreversibly. This is not a surprising result, but we should inquire whether such a response would not occur at a small but non-zero value of this ratio. In fact, (45) indicates that, when a drop of given size and fluid properties is subjected to increasing values of the relative velocity U_∞, the amplitude of the oscillations increases at the same (dimensional) values of the modal frequency. This behaviour suggests that the drop will cease to vibrate and will break up (i.e. deform continuously and irreversibly) owing to nonlinear effects, at some large but finite value of the modified Weber number. An estimate of this 'critical value' of the Weber number, obtained by extending our results to include nonlinear effects, will be presented in a separate paper. It is sufficient for our purposes to cite the broad experimental evidence which clearly indicates that the value of the Bond number, associated with the 'critical' value of the Weber number, is less than 11·2 and corresponds to absolute stability.

The existence of a critical Weber number *We* has been reported by several observers, some of whom detailed the effect of viscosity. For the case of water, Hanson, Domick & Adams (1963) reported critical values between 3·6 and 7·1, Hinze (1948, 1955) reported values between 9 and 13, while Volynski (1948) reported values between 7·2 and 7·3. These observers reported that, when the critical value of the Weber number is approached from below, by increasing the gas velocity relative to a drop of given size and fluid properties, the dynamic response is such that the drop does not vibrate, rather, it deforms continuously into a 'bag-like' configuration that subsequently ruptures. Some very clear photographs of this deformation have been presented by Hanson *et al.* (1963), Anderson & Wolfe (1965) and Simpkins (1971). It should be emphasized that no signs of instability are evident in these photographs. Our conclusion is that, at the lowest level of external gasdynamic flow required to break up the drop, the response is due entirely to aerodynamic effects and not to the unstable growth of surface waves. Further, at large values of the Bond number, where the drop is subject to unstable (exponential) response on the time scale t, it is simultaneously subject to continuous irreversible (algebraic) deformation on the time scale t^*.

6. A composite representation

The results of previous sections indicate that initially, i.e. at values of the time

$$t = O(\epsilon^{*\frac{1}{2}}), \quad t^* = O(1),$$

the effect of acceleration is balanced by the first mode of the aerodynamic pressure distribution, while the remaining pressure modes deform the drop from its initial spherical shape. After a sufficiently long time, however,

$$t = O(1), \quad t^* = O(\epsilon^{*-\frac{1}{2}}),$$

the aerodynamic pressure can no longer balance the effect of acceleration and a self-excitation characteristic of Taylor instability is manifest. Both (45), which describes the initial aerodynamic deformation, and (23), which describes the selective amplification up to a cut-off wavenumber of each mode of the initial deformation, are asymptotic expansions valid for small, but different, orders of the time. These expansions may be combined to form a composite representation (Van Dyke 1964). In terms of the 'outer' variable

$$\tau = \epsilon^{*\frac{1}{2}} t^*, \tag{47}$$

the composite representation is

$$\eta_s = 1 - \frac{\tau^2}{8} \sum_2^\infty n(2n+1) C_n P_n(\chi) \cos\left\{n\left[\frac{(n-1)(n+2)}{Bo} - \chi\right] g^{(1)}\right\}^{\frac{1}{2}} \tau, \tag{48}$$

which, when rewritten in terms of the 'inner' variable t^* and evaluated in the limit $\epsilon^* \to 0$, reproduces the expansion of (45).

The composite expansion (48) provides a means of computing the asymptotic response appropriate to an initial surface pressure distribution (43*b*) with a finite number of modes. In general, such computations have revealed a 'quasi-stable' breakup regime for values of the Bond number between 11·2 and about 10^4, where the small rate of exponential growth confines the effect of instability to a neighbourhood of the windward surface as it distorts algebraically in time. Such behaviour is manifest in the classical photographs of Engle (1958) for values of the Bond number near 10^3. For values of the Bond number above about 10^5, the composite expansion exhibits an initial (algebraic) distortion phase followed by a sudden (exponential) shattering of the windward surface. Photographs of such behaviour at large values of the Bond number have been presented recently by Reinecke & Waldman (1970), who also present x-radiogram evidence of what they term a 'catastrophic mode' of breakup at Bond numbers of 10^5 and 10^6. Furthermore, their empirical correlation of a 'breakup time', which varies as $(Bo)^{-\frac{1}{4}}$, is consistent with our result (26) for the maximum growth rate of unstable waves.

At large values of the Bond number, the number of unstable modes (24) becomes very great. For example, at a Bond number of 10^5, there are 316 unstable modes, the maximum rate of exponential growth occurring with the 183rd mode, equation (25). Therefore, the precise configuration of the shattering drop is extremely sensitive to very small variations in the pressure distribution. Nonetheless, it is instructive to display some of the possible realizations arising from

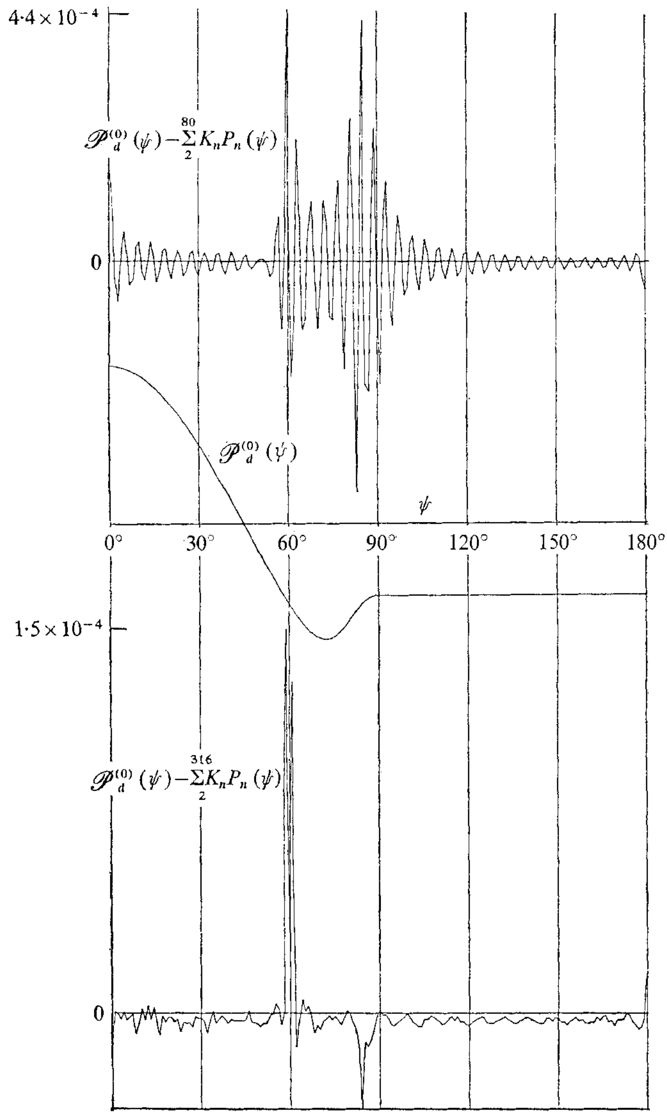

FIGURE 5. Two approximations to a reference surface pressure distribution $\mathscr{P}_d^{(0)}$.

the interference of unstable modes while emphasizing that such realizations are only representative in a statistical sense.

We consider, for the purpose of illustration, the reference pressure distribution $\mathscr{P}_d^{(0)}$ shown in figure 5 and given by a piecewise analytic function which is continuous through all derivatives, except at $\psi = 59°$ and $61°$, where the function is continuous through the second derivative. The intent here is to introduce an undetectable perturbation in the pressure distribution, namely, a large variation in the second derivative between 59° and 61°. The effect of such a perturbation, in the presence of an otherwise smooth pressure distribution, will be displayed by means of the composite expansion.

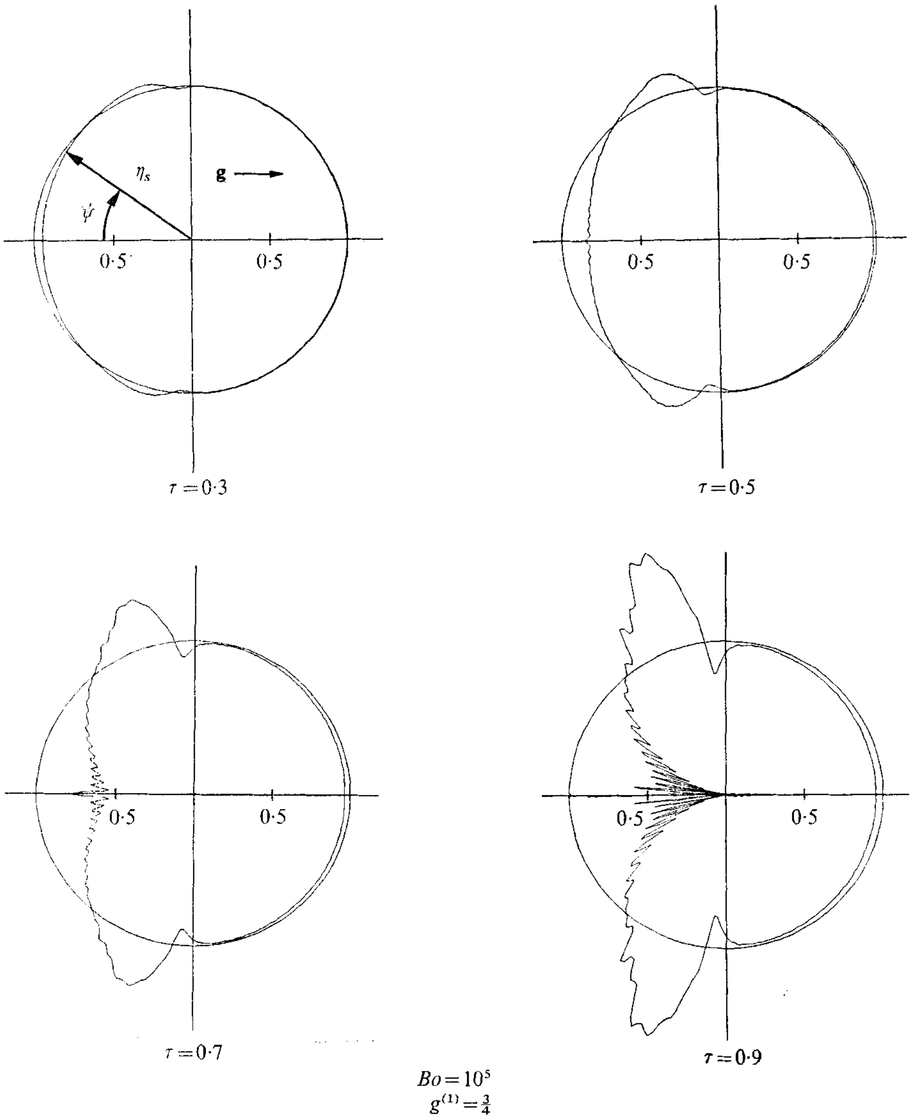

FIGURE 6. An example of the dynamic response at a Bond number of 10^5.

In order to demonstrate the sensitivity of the unstable surface shape to variations in the surface pressure distribution, we consider two approximations to the reference distribution based on the partial sum of 80 and 316 Fourier coefficients, respectively, at a Bond number of 10^5. As shown in figure 5, both partial sums approximate the given function to one part in 10^5 except near 60°, as one would expect. Figures 6 and 7 show the profound effect of differences between high modal coefficients even when the pressure distributions under consideration are identical to four decimal places. In figure 6, all the modal coefficients above 80 have been set identically to zero, and the drop surface manifests considerable structure. In figure 7, all 316 unstable modes have been summed and may be seen

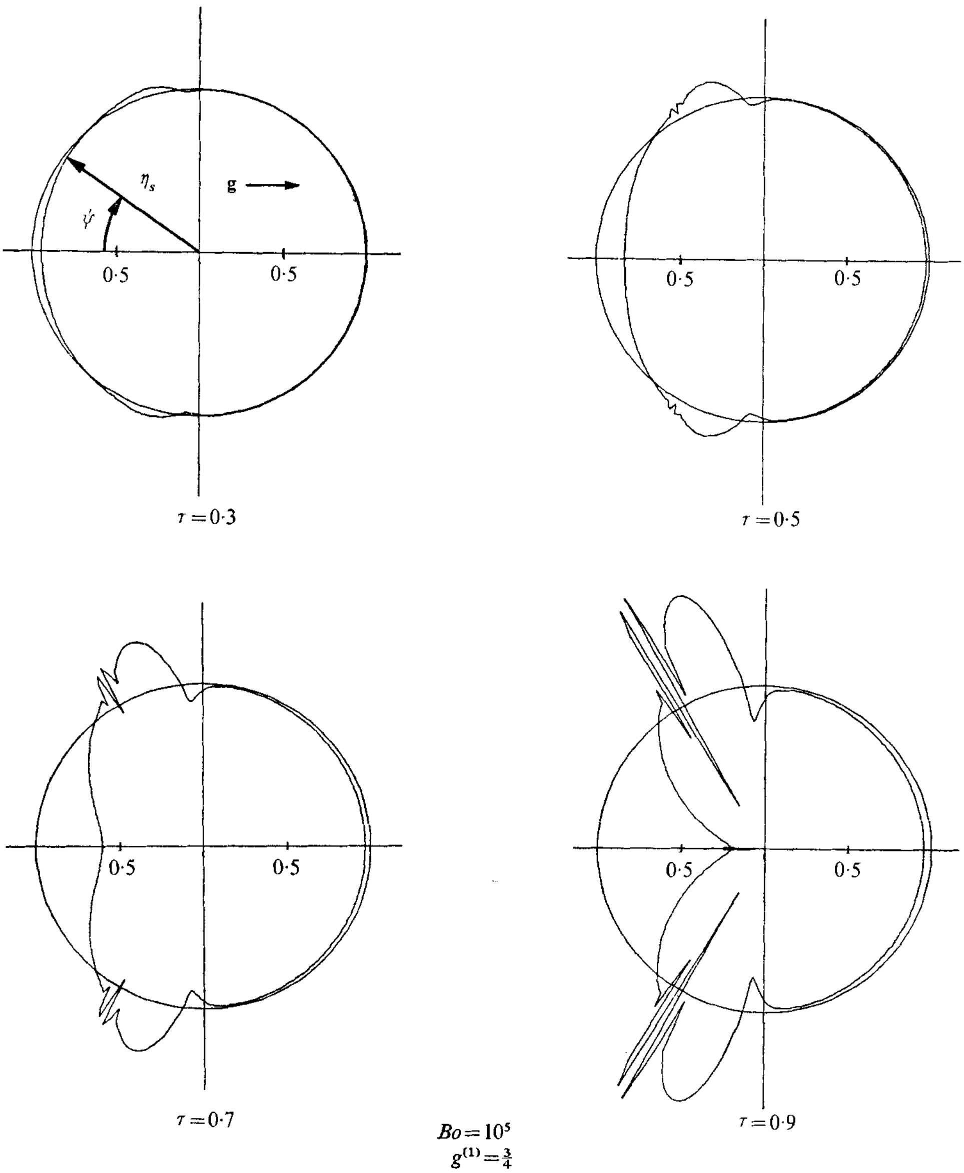

FIGURE 7. An example of the dynamic response at a Bond number of 10^5.

to interfere constructively so as to produce a smooth surface except at 60°, where the reference pressure distribution was intentionally perturbed. Figure 7 bears considerable resemblance to the unstable 'fingers' photographed at a Bond number of 10^5 by Reinecke & Waldman (1970), and reproduced, with their kind permission, in figure 8 (plate 1). It should be emphasized that the pressure distribution appropriate to figure 8 is not known and the comparison with figure 7 is qualitative.

7. Conclusions

We summarize our results by reference to figure 9, which depicts the regimes of the dynamic response of a drop to a suddenly applied pressure distribution in terms of the order of the Weber and Bond numbers and as a function of the time t^* appropriate to the aerodynamic distortion. At values of $We = O(\epsilon^*)$ and below, the drop vibrates for all time. Above the critical value of the Weber number, the response is not vibratory, rather, the drop breaks up by deforming algebraically in time owing to nonlinear effects. The value of the Bond number associated with the critical Weber number is $O(1)$ and is less than 11·2. Therefore, for values of the Weber number slightly above the critical value, the drop is stable to small disturbances (stability refers only to the absence of exponentially growing surface waves) and the breakup is induced by aerodynamic forces. Above the lowest critical value of the Bond number, 11·2, the drop is unstable to small disturbances of arbitrary form. However, until the Bond number reaches a rather large value, say 10^5, the rate of growth of the unstable modes is rather small compared with the aerodynamic deformation. We therefore define a 'quasi-stable' regime in which the deformation is aerodynamically induced (algebraic in time), and the effects of acceleration are manifest as waves on the windward surface. At this point, it is appropriate to mention an effect which is relevant to the breakup of liquid drops and which has not been considered in this study. A survey of the available experimental data for shock wave-water drop interaction indicates that, above a Bond number of about 10^2, liquid is removed from the drop and is visible in the wake region as a spray. At lower values of the relative velocity, there appears to be little or no mass loss and the breakup of the drop results from an irreversible 'bag-type' deformation. In the so-called 'quasi-stable' region, $10^2 < Bo < 10^5$, the deformation is essentially algebraic in time and the mass removal appears to contribute significantly to the disintegration of the drop. A boundary-layer 'stripping' model has been presented by Ranger & Nicholls (1969), in order to account for such mass removal. However, an experimental 'stripping mode' correlation, presented by Reinecke & Waldman (1970), indicates that the mass loss is occurring at a much higher rate than would be accounted for by boundary-layer stripping alone. This discrepancy has been discussed by Collins & Charwat (1971), who have presented a semi-empirical model for intermittent disintegration deriving from capillary waves. However, their assumption that an initially spherical drop deforms as a spherical cap is not supported by our computations involving both aerodynamic and acceleration effects.

At large values of the Bond number, $O(10^5)$ and above, the effects of acceleration (exponential in time) appear at the characteristic time $t^* = O(\epsilon^{*-\frac{1}{2}})$, after which they dominate the breakup by shattering the drop from the windward surface. The detailed shape of the free surface is described by the selective amplification, up to cut-off, of the initial distortion induced by the aerodynamic pressure distribution. The cut-off wavenumber is approximately $Bo^{\frac{1}{2}}$, which implies an extreme sensitivity to the surface pressure distribution. These conclusions are supported by the recent experimental results of Reinecke &

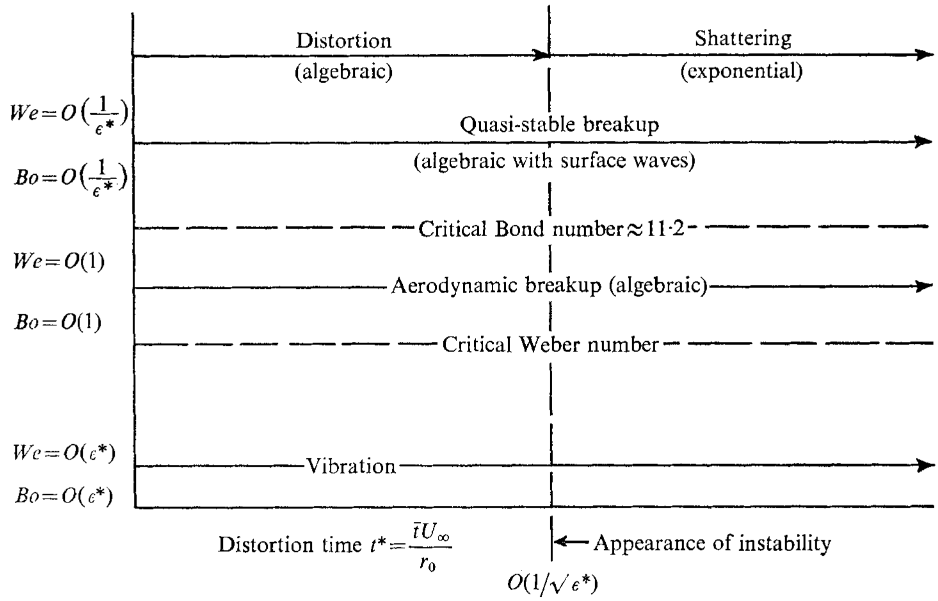

FIGURE 9. Regimes of the dynamic response.

Waldman. At Bond numbers above 10^5, they report, concurrent with their observation of "the onset of large instabilities", a severe departure from the stripping mode, which they describe as "an extremely abrupt, catastrophic disintegration of the drop". They proceed to define a "catastrophic mode" of breakup, the dominant effect of which "is seen to be the rapid growth of surface waves on the windward face of the drop". Further, their empirical correlation of the breakup time in this regime varies with $Bo^{-\frac{1}{4}}$, which is in accord with the maximum rate of growth (26) of unstable acceleration waves. It therefore appears that, above the quasi-stable regime, $Bo > 10^5$, the effect of acceleration controls the breakup of liquid drops through unstable shattering of the windward surface.

Appendix

The kinematic condition at the free surface is approximated by

$$\epsilon\frac{\partial\bar{\eta}^{(1)}(\theta,\bar{t})}{\partial\bar{t}} = \epsilon\frac{\partial\bar{\phi}^{(1)}(\bar{r}_0,\theta,\bar{t})}{\partial\bar{r}} + O(\epsilon^2), \tag{A 1}$$

where, for $\bar{r} \leqslant \bar{r}_0$,

$$\nabla^2\bar{\phi}^{(1)} = 0. \tag{A 2}$$

The hydrodynamic pressure within the drop is given by

$$\frac{\bar{\mathscr{P}}_l}{\bar{\rho}_l} = -\bar{g}\bar{r}\cos\theta - \frac{\partial\bar{\phi}}{\partial\bar{t}},$$

whence, at the surface $\bar{r} = \bar{r}_s$,

$$\bar{\mathscr{P}}_l^{(0)} + \epsilon\bar{\mathscr{P}}_l^{(1)} = -\bar{\rho}_l\left\{\bar{g}\bar{r}_0\cos\theta + \epsilon\left[\bar{g}\bar{\eta}^{(1)}\cos\theta + \frac{\partial\bar{\phi}^{(1)}(\bar{r}_0,\theta,\bar{t})}{\partial\bar{t}}\right]\right\}.$$

The pressure difference across the surface of the drop is given by

$$\bar{\mathscr{P}}_T^{(0)} + \epsilon \bar{\mathscr{P}}_T^{(1)} = \frac{\bar{\sigma}}{\bar{r}_0^2}\left[2\bar{r}_0 - \epsilon\left(2\bar{\eta}^{(1)} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\bar{\eta}^{(1)}}{\partial\theta}\right)\right], \tag{A 3}$$

where $\bar{\sigma}$ is the surface tension coefficient. The hydrodynamic and surface tension stresses are related to the externally applied pressure at the surface

$$\bar{\mathscr{P}}_e = \bar{\mathscr{P}}_e^{(0)} + \epsilon \bar{\mathscr{P}}_e^{(1)} \tag{A 4}$$

through $(2c)$. Application of $(2c)$ to $O(1)$ yields

$$\bar{\mathscr{P}}_e^{(0)} = -\bar{\rho}_l \bar{g}\bar{r}_0 \cos\theta - \frac{2\bar{\sigma}}{r_0}, \tag{A 5}$$

and to $O(\epsilon)$ gives

$$\bar{\mathscr{P}}_e^{(1)} = -\bar{\rho}_l \frac{\partial\bar{\phi}^{(1)}}{\partial\bar{t}} - \bar{\rho}_l \bar{g}\bar{\eta}^{(1)}\cos\theta + \frac{\bar{\sigma}}{\bar{r}_0^2}\left(2\bar{\eta}^{(1)} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\bar{\eta}^{(1)}}{\partial\theta}\right). \tag{A 6}$$

The unit normal vector at the surface of the droplet is given by

$$\mathbf{n} = \frac{\nabla F}{|\nabla F|} = \mathbf{e}_r - \epsilon\bar{\eta}_\theta^{(1)}\mathbf{e}_\theta + O(\epsilon^2), \tag{A 7}$$

where the subscript θ refers to differentiation with respect to that variable. The differential surface area, to $O(\epsilon)$, is

$$d\bar{s} = (\bar{r}_0^2 + \epsilon 2\bar{r}_0\bar{\eta}^{(1)})\sin\theta\, d\phi\, d\theta. \tag{A 8}$$

Application of $(2d)$ to $O(1)$ yields

$$\begin{aligned}\mathbf{F}^{(0)} &= \int_0^\pi\int_0^{2\pi}[\bar{\rho}_l\bar{g}\bar{r}_0\cos\theta - \bar{\sigma}]\, r_0^2\,\mathbf{e}_r\sin\theta\, d\theta \\ &= 2\pi\int_0^\pi \bar{\rho}_l\bar{g}\bar{r}_0^3\,\mathbf{k}\cos^2\theta\sin\theta\, d\theta \\ &= \tfrac{4}{3}\pi\bar{r}_0^3\bar{\rho}_l\bar{g}\mathbf{k} = M\mathbf{g}.\end{aligned} \tag{A 9}$$

Under the assumption of uniform acceleration, we require that

$$\mathbf{F}^{(1)} = -\int_0^\pi\int_0^{2\pi}[2\bar{r}_0\bar{\mathscr{P}}_e^{(0)}\bar{\eta}^{(1)}\mathbf{e}_r - \bar{r}_0\,\mathscr{P}_e^{(0)}\bar{\eta}_\theta^{(1)}\mathbf{e}_\theta + \bar{r}_0^2\mathscr{P}_e^{(1)}\mathbf{e}_r]\sin\theta\, d\phi\, d\theta = 0. \tag{A 10}$$

When the constraint $(2b)$ is imposed, the first term of (A 10) becomes

$$\begin{aligned}-\int_0^\pi\int_0^{2\pi} 2\bar{r}_0\bar{\mathscr{P}}_e^{(0)}\bar{\eta}^{(1)}\mathbf{e}_r\sin\theta\, d\phi\, d\theta \\ &= 4\pi\bar{r}_0\mathbf{k}\int_0^\pi\left[\bar{\rho}_l\bar{g}\bar{r}_0\cos\theta - \frac{\bar{\sigma}}{r_0}\right]\bar{\eta}^{(1)}\sin\theta\cos\theta\, d\theta \\ &= 4\pi\bar{r}_0^2\bar{\rho}_l\bar{g}\mathbf{k}\int_0^\pi \bar{\eta}^{(1)}\sin\theta\cos^2\theta\, d\theta.\end{aligned} \tag{A 11}$$

Application of constraint equations $(2a, b)$ reduces the second term of (A 10) to

$$r_0 \int_0^{\pi}\int_0^{2\pi} \bar{\mathscr{P}}_e^{(0)} \bar{\eta}_\theta^{(1)} \mathbf{e}_\theta \sin\theta \, d\theta$$

$$= -2\pi r_0^2 \mathbf{k} \int_0^{\pi} \bar{\mathscr{P}}_e^{(0)} \bar{\eta}_\theta^{(1)} \sin^2\theta \, d\theta = 2\pi r_0^2 \mathbf{k} \int_0^{\pi} \bar{\eta}^{(1)} \frac{\partial}{\partial\theta} [\mathscr{P}_e^{(0)} \sin^2\theta] \, d\theta$$

$$= -6\pi \bar{r}_0^2 \bar{\rho}_l \bar{g} \mathbf{k} \int_0^{\pi} \bar{\eta}^{(1)} \sin\theta \cos^2\theta \, d\theta. \qquad \text{(A 12)}$$

We select a form for $\bar{\mathscr{P}}_e^{(1)}(\theta, \bar{t})$, namely,

$$\bar{\mathscr{P}}_e^{(1)} = \mathscr{C} \bar{\rho}_l \bar{g} \cos\theta, \qquad \text{(A 13)}$$

whence, by (A 10–13),

$$\mathscr{C} = -\frac{3}{2} \int_0^{\pi} \bar{\eta}^{(1)} \sin\theta \cos^2\theta \, d\theta$$

$$= -\int_0^{\pi} \bar{\eta}^{(1)} P_2(\cos\theta) \sin\theta \, d\theta, \qquad \text{(A 14)}$$

where $P_n(\cos\theta)$ denotes the zonal harmonic of order n.

In summary, the governing equations, to $O(\epsilon)$, are

$$\bar{\rho}_l \frac{\partial \bar{\phi}^{(1)}}{\partial \bar{t}} + \bar{\rho}_l \bar{g} \bar{\eta}^{(1)} \cos\theta = \frac{\bar{\sigma}}{\bar{r}_0^2} (2 + L) \bar{\eta}^{(1)} - \bar{\rho}_l \bar{g} \mathscr{C} \cos\theta \qquad \text{(A 15)}$$

on $\bar{r} = \bar{r}_0$, where

$$L = \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta},$$

$$\mathscr{C} = -\int_0^{\pi} \bar{\eta}^{(1)} P_2(\cos\theta) \sin\theta \, d\theta,$$

and

$$\frac{\partial \bar{\eta}^{(1)}(\theta, \bar{t})}{\partial \bar{t}} = \frac{\partial \bar{\phi}^{(1)}(\bar{r}_0, \theta, \bar{t})}{\partial \bar{r}}, \qquad \text{(A 16)}$$

$$\nabla^2 \bar{\phi}^{(1)} = 0 \quad (\bar{r} \leqslant \bar{r}_0), \qquad \text{(A 17)}$$

with the constraints

$$\int \bar{\eta}^{(1)} d\bar{s} = \int \bar{\eta}^{(1)} \cos\theta \, d\bar{s} = 0. \qquad \text{(A 18)}$$

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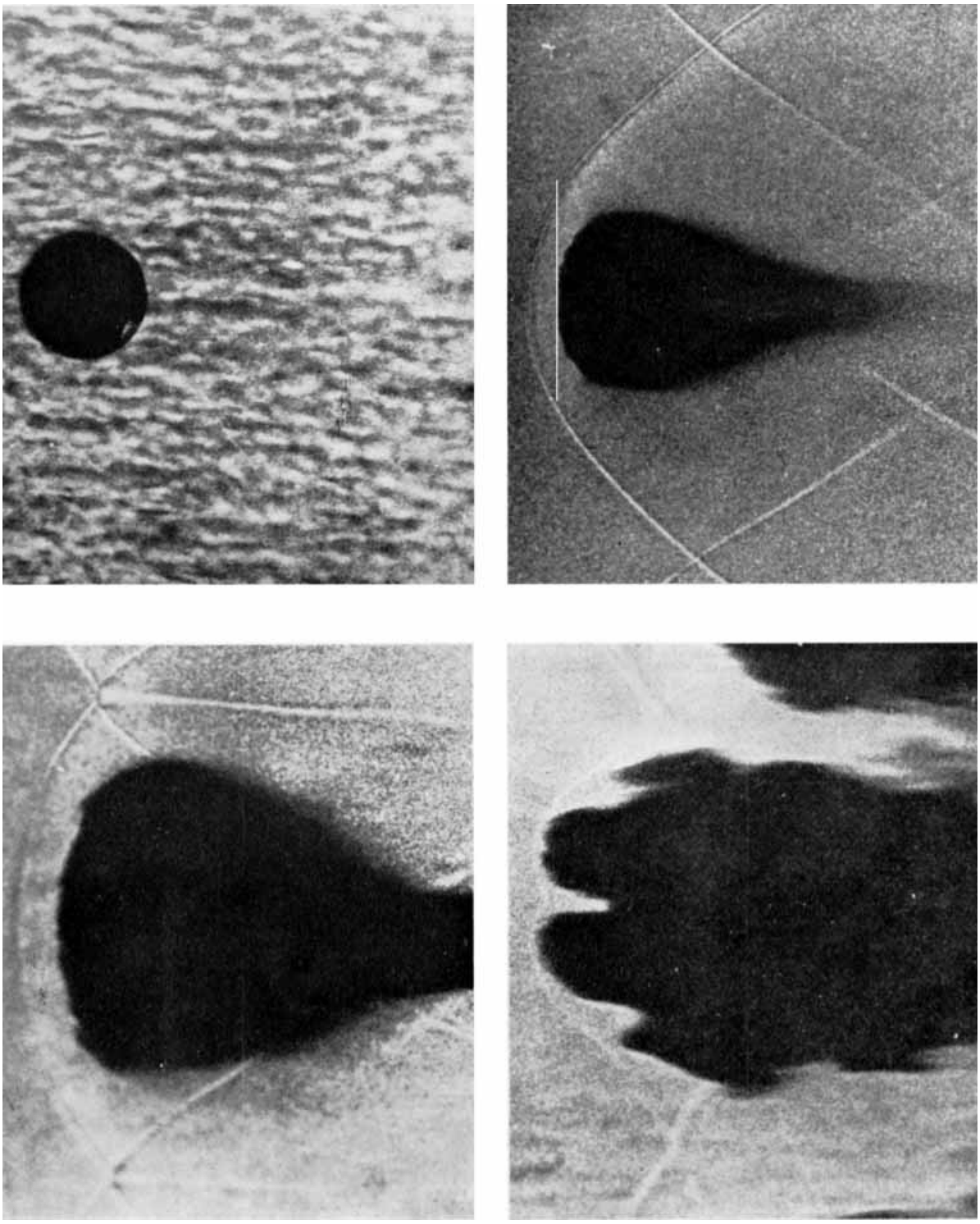

FIGURE 8. Unstable shattering of the windward surface.
Reproduced from Reinecke & Waldman (1970).